

A THEOREM ON THE LIGHTEST GLUEBALL STATE

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ABSTRACT

This paper is devoted to proving that, in QCD, the lightest glueball state must be the scalar with $J^{PC} = 0^{++}$. The proof relies upon the positivity of the path integral measure in Euclidean space and the fact that interpolating fields for all spins can be bounded by powers of the scalar glueball operator. The problem presented by the presence of vacuum condensates is circumvented by considering the time evolution of the propagators and exploiting the positivity of the Hamiltonian.

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In this paper I shall show that, if glueball states exist, then the lightest one must be the 0^{++} scalar. There has recently been a renewed flurry of interest, both experimental and theoretical, in these very interesting states and the situation is beginning to clarify [1]–[5]. In spite of this, the situation still remains unresolved and somewhat ambiguous so exact results such as that presented here are of some interest. Much detailed analysis has now been performed on a large amount of recent experimental data with the result that a few rather good candidates have emerged particularly in the region 1.5-1.7GeV [1]. Potential, bag [4] and instanton gas [2] models suggest that the lowest state should be a scalar and that its mass should be in the above range. All of these models, in spite of having the virtue of incorporating the correct low energy physics of QCD, are only effective representations of the full theory, and so their accuracy is difficult to evaluate. However, recent lattice simulations of QCD based on an extensive amount of data are in general agreement with these model results [3]. On the other hand, estimates from a field theoretic model [5] indicate that the 2^{++} tensor should be the lightest state whereas a QCD sum rule analysis indicates that it should be the 0^{-+} pseudoscalar [6]. This disagreement between the QCD sum rules and the lattice measurements is somewhat surprising since they ought to be the least model dependent and therefore the most reliable. However, the lattice simulations use a quenched, or valence, approximation, which is not generally believed to be a major source of error, and the QCD sum rules have difficulty satisfying a low energy theorem. In any case, as already stated above, the claim of this paper is that, regardless of the model or approximation used, the scalar must be the lightest glueball state. I shall now show why this must be true.

To begin I shall first review some standard formalism as it applies to scalar and pseudoscalar glueballs before generalizing to arbitrary states. These spinless states can be described by the operators:

$$G(x) = f_G F_{\mu\nu}^a(x) F_a^{\mu\nu}(x) \quad \text{and} \quad \tilde{G}(x) = f_{\tilde{G}} F_{\mu\nu}^a(x) \tilde{F}_a^{\mu\nu}(x) \quad (1)$$

where $\tilde{F}_a^{\mu\nu}(x) \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}^a(x)$ is the dual field tensor and f_G and $f_{\tilde{G}}$ are constants. The scalar correlator

$$\Gamma(\mathbf{x}, t) \equiv \langle 0 | T[G(\mathbf{x}, t) G(0)] | 0 \rangle \quad (2)$$

has a standard path integral representation:

$$\Gamma(\mathbf{x}, t) = \int \mathcal{D}A_\mu^a e^{\frac{i}{4} \int F_{\mu\nu}^a F_a^{\mu\nu} d^4x} \det(\not{D} + m) G(\mathbf{x}, t) G(0) \quad (3)$$

A sum over quark flavors is to be understood. By inserting a complete set of states $|N\rangle$ this can also be written

$$\Gamma(\mathbf{x}, t) = \sum_N |\langle 0|G(0)|N\rangle|^2 e^{(iE_N t - i\mathbf{p}_N \cdot \mathbf{x})} \theta(t) + (t \rightarrow -t) \quad (4)$$

from which a corresponding Kallen-Lehmann representation can be inferred.

A useful subsidiary quantity to consider is (for $t > 0$)

$$Q(t) \equiv \int d^3x \Gamma(\mathbf{x}, t) \quad (5)$$

$$= \sum_N |\langle 0|G(0)|N\rangle|^2 \delta^{(3)}(\mathbf{p}_N) e^{iM_N t} \quad (6)$$

where M_N is the invariant mass of the state $|N\rangle$. The Euclidean version of this (effectively given by taking $t \rightarrow i\tau$) implies that, when $\tau \rightarrow \infty$,

$$Q_E(\tau) \equiv Q(i\tau) \approx e^{-M_0 \tau} \quad (7)$$

where M_0 is the mass of the lightest contributing state. An analogous result can be derived for $\Gamma(\mathbf{x}, t)$ via its Kallen-Lehmann representation where the exponential decay arises from the large τ or $|x|$ behaviour of the free Feynman propagator. There are a couple of points worth remarking about this before proceeding. First, in pure QCD, where the $0^{\pm+}$ glueballs are expected to be the lightest states in their respective channels, $M_0 = M_G$ or $M_{\tilde{G}}$. In the full theory, however, the lightest states are those of 2 pions and 3 pions, respectively, and the glueballs become unstable resonances and mix with quark states. In that case $M_0 = M_{2\pi}$ or $M_{3\pi}$. On the other hand, in the limit when τ becomes large, but remains smaller than $\sim 2M_G/\Gamma_G^2$, where Γ_G is the width of the resonance, it can be shown that the exponential decay law, eq. (7), still remains valid but with a mass M_0 given by M_G rather than $M_{2\pi}$; (a similar result obviously also holds for the pseudoscalar case). The point is that, if there are well-defined narrow resonant states present in a particular channel, then they can be sampled by sweeping through an appropriate range of asymptotic τ values where they dominate, since τ is conjugate to M_N [7].

The basic inequality that we shall employ is that, in Euclidean space,

$$(F_{\mu\nu}^a \pm \tilde{F}_a^{\mu\nu})^2 \geq 0 \quad \Rightarrow \quad f_G^{-1} G(x) \geq f_{\tilde{G}}^{-1} \tilde{G}(x) \quad (8)$$

Although this inequality holds for classical fields, it can be exploited in the quantized theory by using the path integral representation, eq. (3), in Euclidean space where the

measure is positive definite. The positivity of the measure has been skillfully used by Weingarten [8] to prove that in the quark sector the pion must be the lightest state. Here, when combined with the inequality (8), it immediately leads to the inequalities (valid for $\tau > 0$)

$$f_G^{-2}\Gamma_E(\mathbf{x}, \tau) \geq f_{\tilde{G}}^{-2}\tilde{\Gamma}_E(\mathbf{x}, \tau) \quad \text{and} \quad f_G^{-2}Q_E(\tau) \geq f_{\tilde{G}}^{-2}\tilde{Q}_E(\tau) \quad (9)$$

By taking τ large (but $< 2M_G/\Gamma_G^2$) and using (7), the inequality

$$M_G \leq M_{\tilde{G}} \quad (10)$$

easily follows. In pure QCD where these glueballs are isolated singularities, their widths vanish and the limit $\tau \rightarrow \infty$ can be taken without constraint.

Although this is the result we want, its proof presumes the absence of a vacuum condensate $E \equiv \langle 0|G(0)|0\rangle$. It is generally believed that $E \neq 0$ so the lightest state contributing to the unitarity sum in eq. (4) is, in fact, the vacuum in which case $M_0 = 0$ and the large τ behaviour of $\Gamma_E(\mathbf{x}, \tau)$ is a constant, E^2 , rather than an exponential. Thus, the inequalities (9) are trivially satisfied for asymptotic values of τ since there is no condensate in the pseudoscalar channel. To circumvent this problem it is prudent to consider the time evolution of either $Q(t)$ or $\Gamma(\mathbf{x}, t)$ since this removes the offending condensate contribution. For example, (for $t > 0$)

$$\frac{dQ(t)}{dt} = \sum_N |\langle 0|G(0)|N\rangle|^2 \delta^{(3)}(\mathbf{p}_N) iM_N e^{iM_N t} \quad (11)$$

The vacuum state clearly does not contribute to this so, in Euclidean space, the large τ behaviour of $\dot{Q}_E(\tau)$ is, up to a factor $-M_0$, just that of eq. (7). Now, (for $t > 0$), consider the following:

$$\frac{\partial \Gamma(\mathbf{x}, t)}{\partial t} = \langle 0 | \frac{\partial G(\mathbf{x}, t)}{\partial t} G(0) | 0 \rangle \quad (12)$$

$$= \langle 0 | i[H, G(\mathbf{x}, t)] G(0) | 0 \rangle \quad (13)$$

$$= -i \langle 0 | G(\mathbf{x}, t) H G(0) | 0 \rangle \quad (14)$$

where, in the last step, the condition $H|0\rangle = 0$ has been imposed. At the classical level H is positive definite. We can therefore repeat our previous argument by working in Euclidean space and combining the inequalities (8) with a path integral representation for (14) to obtain (for $\tau > 0$) the inequalities

$$f_G^{-2} \frac{\partial \Gamma_E(\mathbf{x}, \tau)}{\partial \tau} \geq f_{\tilde{G}}^{-2} \frac{\partial \tilde{\Gamma}_E(\mathbf{x}, \tau)}{\partial \tau} \quad \text{and} \quad f_G^{-2} \dot{Q}_E(\tau) \geq f_{\tilde{G}}^{-2} \dot{\tilde{Q}}_E(\tau) \quad (15)$$

The large τ limit then leads to

$$f_G^{-2} M_G e^{-M_G \tau} \geq f_{\tilde{G}}^{-2} M_{\tilde{G}} e^{-M_{\tilde{G}} \tau} \quad (16)$$

from which (10) follows even in the presence of condensates [9].

Although this argument is essentially correct, it still remains incomplete in that we need to clarify the nature of the path integral representation for (14). The point is that the Hamiltonian, H , that generates time translations must be expressed in terms of canonical momentum and co-ordinate field variables. Thus, a path integral representation for (14) must first be written in Hamiltonian form; unfortunately, however, the resulting measure is not necessarily positive definite even in the Euclidean region. Thus, for our above argument to be valid we need to show that after integrating the Hamiltonian form over canonical momenta, the classical expressions for the operators can still be used and that, in Euclidean space, the resulting measure of the Lagrangian form remains positive. I shall first sketch how this comes about in a quantum mechanical context before generalizing to field theory. Some of the subtleties encountered are closely related to (normal) ordering problems that arise when dealing with operators which, like H , depend on both P , the canonical momentum and Q , the canonical coordinate [10]. Consider, first, the quantum mechanical analog of (2):

$$\Gamma(t) \equiv \langle 0 | T[G(t)G(0)] | 0 \rangle \quad (17)$$

By analogy with eq. (1) the operator G is to be considered a function of Q . A path integral representation for this can be generated using the standard procedure of dividing up the infinite time interval into discrete infinitesimal sequences of size ϵ and, at each discrete time t_n , say, exploiting the completeness of momentum and coordinate eigenstates (labelled by p_n and q_n , respectively) [10]:

$$\Gamma(t) = \int \frac{dp_1}{2\pi} \dots \int \frac{dp_{N+1}}{2\pi} \int dq_1 \dots \int dq_N e^{i \sum_{n=1}^{N+1} [p_n(q_n - q_{n-1}) - H(p_n, q_n)\epsilon]} G(q_k) G(q_l) \quad (18)$$

Here, $\epsilon = t_{n+1} - t_n$ and k and l are defined such that $t_k = t$ and $t_l = 0$. It is understood that the limits $N \rightarrow \infty$ and $\epsilon \rightarrow 0$ are to be taken in such a way that the total time interval $N\epsilon \rightarrow i\infty$ in order to pick out the ground state expectation value [10]. An analogous expression for $d\Gamma(t)/dt$ can be derived in a similar fashion:

$$\frac{d\Gamma(t)}{dt} = \int \frac{dp_1}{2\pi} \dots \int \frac{dp_{N+1}}{2\pi} \int dq_1 \dots \int dq_N e^{i \sum_{n=1}^{N+1} [p_n(q_n - q_{n-1}) - H(p_n, q_n)\epsilon]} G(q_k) H(p_m, q_m) G(q_l) \quad (19)$$

Here m is defined such that $t_m \subseteq [t_k, t_l]$. This freedom in the choice of t_m is simply a reflection of the time-invariance of the operator H . If t_m lies outside of this domain, then the integral will vanish since one can then “undo” the path integral using the completeness relations and move the operator H to later and later times (or earlier and earlier ones, as appropriate) until it annihilates on the vacuum. A similar procedure can be used to verify that this same expression, eq. (19), can be derived by directly differentiating the right-hand-side of eq. (18) with respect to $t_k(=t)$. This brings down a factor $i[H(p_{k+1}, q_{k+1}) - H(p_k, q_k)]$ leading to the path integral manifestation of eq. (13). However, depending on the time ordering, one of these factors of H can be moved (again using completeness to “undo” the path integral) so that it eventually acts on the vacuum and vanishes. In this way, one can verify that the operator manipulations going from eqs. (12) - (14) are faithfully reproduced by the path integral and that the time evolution of $\Gamma(t)$ is given by eq. (19).

Typically, and in particular in QCD, H is quadratic in P so the integrals over the p_n in eq. (18) are simple gaussians which straightforwardly lead to the conventional Lagrangian form (up to an overall vacuum-vacuum amplitude normalization constant):

$$\Gamma(t) = \int dq_1 \dots \int dq_N e^{i \sum_{n=1}^{N+1} L(\dot{q}_n, q_n) \epsilon} G(q_k) G(q_l) \quad (20)$$

Here $\dot{q}_n \equiv (q_n - q_{n-1})/\epsilon$. When elevated to field theory eq. (20) becomes eq. (3). A similar set of manipulations can be carried out on eq. (19). Now, however, there is an added complication since H occurs not only in its conventional place in the exponent of the measure, but also in the integrand itself. Thus, in addition to the usual gaussian integral, its counterpart weighted with $\frac{1}{2}p_n^2$ is also needed. Upon integration this leads to a factor $(\frac{1}{2}\dot{q}_n^2 + H_0)$ where H_0 is a (positive) constant $\sim \epsilon^{-2}$. This background energy can be removed by a redefinition of $H(P, Q)$ which only changes the energy of the vacuum, E_0 , without affecting mass differences [9]. With this definition, (19) becomes

$$\frac{d\Gamma(t)}{dt} = \int dq_1 \dots \int dq_N e^{i \sum_{n=1}^{N+1} L(\dot{q}_n, q_n) \epsilon} G(q_k) H(\dot{q}_m, q_m) G(q_l) \quad (21)$$

As a check on the manipulations leading to this, notice that it is consistent with the direct differentiation of (20) which was normalized to the vacuum-vacuum amplitude. The Euclidean field theoretic version of (21) can now be used to justify the inequalities leading to our central result eq. (10).

The extension of the above argument to the general case showing that the scalar must be lighter than all other glueball states, can now be effected. Introduce an operator, $T_{\mu\nu\alpha\beta\dots}(x)$, constructed out of a sufficiently long string of $F_{\mu\nu}^a(x)$'s and $\tilde{F}_a^{\mu\nu}(x)$'s that it can, in principle, create an arbitrary physical glueball state of a given spin. Generally speaking a given T once constructed can, of course, create states of many different spins, depending on the details of exactly how it is constructed. As a simple example consider the fourth-rank tensor [11]

$$T_{\mu\nu\alpha\beta}(x) = F_{\mu\nu}(x)F_{\alpha\beta}(x) \quad (22)$$

which creates glueball states with quantum numbers 2^{++} and 0^{++} . Now, in Euclidean space, the magnitude of any component of $F_{\mu\nu}^a(x)$, or $\tilde{F}_a^{\mu\nu}(x)$, is bounded by the magnitude of $[F_{\mu\nu}^a(x)F_a^{\mu\nu}(x)]^{\frac{1}{2}}$. Hence, any single component of $T_{\mu\nu\alpha\beta}(x)$ must, up to a constant, be bounded by $G(x)$:

$$T_{\mu\nu\alpha\beta}(x) \leq f_G^{-1}G(x) \quad (23)$$

This inequality is the analog of (8) and so the same line of reasoning used to exploit that inequality when proving (10) can be used here. Following the same sequence of steps leads to the conclusion that M_G must be lighter than the lightest state interpolated by $T_{\mu\nu\alpha\beta}(x)$, from which the inequality

$$M(2^{++}) \geq M(0^{++}) \equiv M_G \quad (24)$$

follows. It is worth pointing out that the pseudoscalar analog of this operator can be similarly bounded thereby leading to the inequality $M(2^{++}) \leq M(2^{-+})$. This argument can be generalized to an arbitrary $T_{\mu\nu\alpha\beta\dots}(x)$ since, again up to some overall constants analogous to f_G of eq. (1), it is bounded by some power (p) of $G(x)$; i.e., for any of its components, $T_{\mu\nu\alpha\beta\dots}(x) \leq G(x)^p$. Now, the operator $G(x)^p$ has the same quantum numbers as $G(x)$ and so can also serve as an interpolating field for the creation of the scalar glueball. The same arguments used to prove that this 0^{++} state is lighter than either the 0^{+-} or the 2^{++} can now be extended to the general case showing that it must be lighter than *any* state created by *any* T ; in other words, the scalar glueball must indeed be the lightest glueball state.

Finally, we make some brief remarks about the conditions under which the bound is saturated. Clearly the inequalities (8) become equalities when $F_{\mu\nu}^a(x) = \tilde{F}_{\mu\nu}^a(x)$ which is also the condition that minimizes the action and signals the dominance of pure non-perturbative instantons. In such a circumstance the 0^{++} and 0^{+-} will be degenerate. However, the proof of the mass inequality (10) only required (8) to be valid at asymptotic

values of $|x|$. Thus, the saturation of this bound actually only rests on the weaker condition that F be self-dual in the asymptotic region where it must vanish like a pure gauge field. Similarly, the saturation of the general inequality showing the scalar to be the lightest state occurs when *all* components of $F_{\mu\nu}^a(x)$ have the same functional dependence at asymptotic values of $|x|$. Although this is a stronger condition than required by the general asymptotic self-dual condition, it is, in fact, satisfied by the explicit single instanton solution. Thus, the splitting of the levels is determined by how much the asymptotic behaviour of the non-perturbative fields differ from those of pure instantons. This therefore suggests a picture in which the overall scale of glueball masses is set by non-perturbative effects driven by instantons (which produce the confining long-range force) but that the level splittings are governed by perturbative phenomena.

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